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On the New Variational Principles and Duality for Periodic Solutions of Lagrange Equations with Superlinear Nonlinearities

Andrzej Nowakowski and Andrzej Rogowski

Faculty of Mathematics, University of Łódź, Banacha 22, 90-238 Łódź, Poland
E-mail: annowako@math.uni.lodz.pl, arogow@math.uni.lodz.pl

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1. INTRODUCTION

We investigate the nonlinear problem

$$\frac{d}{dt}L_{x'}(t, x'(t)) + V_x(t, x(t)) = 0, \qquad \text{a.e. in } R \tag{1.1}$$

$$x(t+T) = x(t),$$

where

(H). *$T > 0$ is arbitrary, and $L, V\colon R \times R^n \to R$ are convex, Gateaux differentiable in the second variable, T-periodic and measurable in t functions.*

We are looking for solutions of (1.1) that are a pair (x, p) of periodic absolutely continuous functions $x, p\colon [0, T] \to R^n$, $x(0) = x(T)$, $p(0) = p(T)$ such that

$$\frac{d}{dt}p(t) + V_x(t, x(t)) = 0,$$
$$p(t) = L_{x'}(t, x'(t)).$$

Of course, if $L(t, x') = \frac{1}{2}|x'|^2$ or $t \to L_p^*(t, p(t))$ (L^* denotes the Fenchel conjugate of $L(t,\cdot)$) is an absolutely continuous function, then our solution of (1.1) belongs to $C^{1,+}([0, T], R^n)$ of continuously differentiable functions x whose derivatives x' are absolutely continuous. In the sequel we assume

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that V_x is superlinear. It is clear that (1.1) is the Euler–Lagrange equation for the functional

$$J(x) = \int_0^T (-V(t, x(t)) + L(t, x'(t))\, dt \tag{1.2}$$

considered on the space A_p of absolutely continuous T-periodic functions x: $R \to R^n$.

Periodic problem (1.1) was studied in the 1980s by many authors for the sublinear as well as for the superlinear case (see, e.g., [6]). However, we believe that our paper may contribute some new look at this problem. This is because we propose to study (1.1) by duality methods in a way that is, to some extent, analogous to the methods developed for (1.1) in sublinear cases [2, 3, 6, 8]. Some cases of (1.1) for superlinear V_x were studied in [1, 5, 7]. It is interesting that the method developed in [5] is based on the dual variational method for the problem, according to the idea discovered by Clarke. Since functional (1.2) is, in general, unbounded in A_p (especially in the superlinear case), it is obvious that we must look for critical points of J of the "minmax" type. The main difficulties that appear here are determining the kind of sets we should choose over which we wish to calculate the "minmax" of J and then linking this value with critical points of J. Of course, we have the mountain pass theorems, the saddle point theorems, the Morse theory (see, e.g., [4, 6, 9]), but none of these exhaust all critical points of J.

Our aim is to find a nonlinear subspace X of A_p defined by the type of nonlinearity of V (and, in fact, L). To be more precise, let us state the basic hypothesis we need:

(H1). *There exist* $0 < \alpha_1 \le \alpha_2$, *and* $d_1, d_2 \in R$ *such that for* $x' \in L^2$

$$d_1 + \frac{\alpha_1}{2}\|x'\|_{L^2}^2 \le \int_0^T L(t, x'(t))\, dt \le \frac{\alpha_2}{2}\|x'\|_{L^2}^2 + d_2. \tag{1.3}$$

$L(t, \cdot)$ *is strictly convex. There exist* $0 < \beta_1 < \beta_2, q_1 > 1, q > 2, k_1, k_2 \in R$ *such that for* $x \in L^q$

$$k_1 + \frac{\beta_1}{q_1}\|x\|_{L^{q_1}}^{q_1} \le \int_0^T V(t, x(t))\, dt \le \frac{\beta_2}{q}\|x\|_{L^q}^q + k_2. \tag{1.4}$$

Having fixed the type of nonlinearities of L and V, we are able to define nonlinear subspaces $\overline{X}$, $\widetilde{X}$, and X as follows. First we put

$$\overline{X} = \left\{ v \in A_0 \colon \frac{\beta_2}{q}\|v\|_{L^q}^q \le \frac{\alpha_1}{4}\|v'\|_{L^2}^2 \right\},$$

where A_0 is the space of absolutely continuous functions v: $[0, T] \to R^n$ with $v' \in L^2$ and $v(0) = 0$. We reduce the space $\overline{X}$ to the set

$$\widetilde{X} = \{ v \in \overline{X} \colon p(t) = L_{x'}(t, v'(t)),\ t \in [0, T],\ \text{belongs to } A \},$$

where A is the space of absolutely continuous functions v: $[0, T] \to R^n$ with $v' \in L^2$ and next to the set $X \subset \widetilde{X}$ with the property that for each $v \in X$ and $c_v \in R^n$, where c_v is a minimizer for the functional $R^n \ni c \to \int_0^T V(t, v(t) + c)\,dt$ (such a minimizer, by (H) and (H1), certainly exists), there exists (possibly another) $\tilde{v} \in X$ such that $V_x(t, v(t) + c_v) = -\frac{d}{dt} L_{x'}(t, \tilde{v}'(t))$, for a.e. $t \in [0, T]$. We assume that X contains at least one element x with $x(0) = 0 = x(T)$.

It is clear that the set X is much smaller than $\widetilde{X}$ and that it depends strongly on the type of nonlinearities V and L. We easily see that X is not in general a closed set in A. As the dual set to X we shall consider the set

$$X^d = \{p \in A_T\colon \text{ there exist } v \in X \text{ and } d_p \in R^n \\ \text{such that } p(t) = L_{x'}(t, v'(t)) - d_p,\ t \in [0, T] \text{ a.e.}\},$$

where A_T denotes the space of absolutely continuous functions v: $[0, T] \to R^n$ with $v' \in L^2$ and $v(T) = 0$. Taking into account the structure of the space X, we shall study the functional

$$J(x, c) = \int_0^T (-V(t, x(t) + c) + L(t, x'(t)))\,dt + l(x(T))$$

on the space $X \oplus R^n$ instead of (1.2) on the space A_p, where

$$l(a) = \begin{cases} 0 & \text{if } a = 0, \\ +\infty & \text{if } a \neq 0. \end{cases}$$

We shall look for a "min" of J over the set X, i.e., actually

$$\min_{x \in X} \max_{c \in R^n} J(x, c).$$

To show that element $\bar{x} \in X$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D, described in the next section. Because of the duality theory we are able to avoid in our proof of the existence of critical points the deformation lemmas, the Ekeland variational principle, or PS-type conditions. One more advantage of our duality results is obtaining for the first time in the superlinear case a measure of a duality gap between the primal and dual functional for approximate solutions to (1.1) (for the sublinear case see [8]).

The main result of our paper is the following:

THEOREM (Main). *If $X \neq \emptyset$ then under hypotheses* (*H*) *and* (*H*1) *there exists a pair* $(\bar{x} + c_{\bar{x}}, \bar{p} + d_{\bar{p}})$, $\bar{x} \in X$, $\bar{p} \in X^d$, $c_{\bar{x}} \in R^n$, $d_{\bar{p}} \in R^n$ *being a solution to* (1.1), *and such that*

$$J(\bar{x}, c_{\bar{x}}) = \min_{x \in X} \max_{c \in R^n} J(x, c) = \min_{p \in X^d} \max_{d \in R^n} J_D(p, d) = J_D(\bar{p}, d_{\bar{p}}).$$

We see that our hypotheses on L and V concern only the convexity of $L(t,\cdot)$ or $V(t,\cdot)$ and that the latter function is of the superquadratic type. We do not assume that $V(t,x)\geq 0$. However, we require that the above set X is nonempty, which we must check in each concrete type of equation. At the end of the paper we give some routines for the equation

$$x''+V_x(t,x)=0.$$

2. DUALITY RESULTS

To obtain a duality principle we need a kind of perturbation of J. Thus define for each $x\in X$ the perturbation of J as

$$J_x(a,y)=\int_0^T (V(t,x(t)+c_x+y(t))-L(t,x'(t)))\,dt-l(x(T)+a) \quad (2.1)$$

for $y\in L^2$, $a\in R^n$. Of course, $J_x(0,0)=-J(x,c_x)$. For $x\in X$ and $p\in X^d$, $d\in R^n$ we define a type of conjugate of J by

$$J_x^{\#}(p,d)=\sup_{y\in L^2}\sup_{c}\left\{\int_0^T\langle y(t),p'(t)\rangle dt-\int_0^T V(t,x(t)+c+y(t))\,dt\right\}$$
$$+\int_0^T L(t,x'(t))\,dt+\inf_{a\in R^n}\{\langle a,d\rangle+l(x(T)+a)\}.$$

By a direct calculation we obtain

$$\begin{aligned}J_x^{\#}(p,d)&=\sup_c\left\{-\langle x(T),d\rangle-\int_0^T\langle x(t)+c,p'(t)\rangle\,dt+\int_0^T L(t,x'(t))dt\right.\\&\qquad\left.+\int_0^T V^*(t,-p'(t))dt\right\}\\&=\sup_c\{-\langle c,p(0)\rangle\}+\int_0^T\langle x'(t),p(t)+d\rangle dt\\&\quad+\int_0^T L(t,x'(t))dt+\int_0^T V^*(t,-p'(t))dt\\&=\int_0^T\langle x'(t),p(t)+d\rangle\,dt+\int_0^T L(t,x'(t))dt\\&\quad+\int_0^T V^*(t,-p'(t))dt+l(p(0)).\end{aligned} \quad (2.2)$$

Now we take "min" from $J_x^{\#}(p,d)$ with respect to $x\in X$ and calculate it. Because X is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this end we use the special structure of the set X^d. First we observe that for each $p \in X^d$ and appropriate d_p there exists $x_p \in X$ such that

$$p(t) + d_p = L_{x'}(t, x'_p(t)),$$

and, by the classical convex analysis argument,

$$x'_p(t) = L^*_p(t, p(t) + d_p),$$

where L^* is a Fenchel conjugate to L. Therefore,

$$\int_0^T \langle x'_p(t), p(t) + d_p \rangle dt - \int_0^T L(t, x'_p(t))\, dt = \int_0^T L^*(t, p(t) + d_p)\, dt.$$

Next let us note that, on the other hand,

$$\begin{aligned} &\int_0^T \langle x'_p(t), p(t) + d_p \rangle dt - \int_0^T L(t, x'_p(t))\, dt \\ &\qquad \le \sup_{x \in X} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle dt - \int_0^T L(t, x'(t))\, dt \right\} \\ &\qquad \le \sup_{x' \in L^2} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle dt - \int_0^T L(t, x'(t))\, dt \right\} \\ &\qquad = \int_0^T L^*(t, p(t) + d_p)\, dt, \end{aligned}$$

and actually all inequalities above are equalities. Therefore we can calculate for $p \in X^d$ and appropriate d_p,

$$\begin{aligned} \sup_{x \in X} -J^{\#}_x(-p, -d_p) &= \sup_{x \in X} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle dt - \int_0^T L(t, x'(t))\, dt \right\} \\ &\qquad - \int_0^T V^*(t, -p'(t))\, dt - l(p(0)) \\ &= \int_0^T L^*(t, p(t) + d_p)\, dt \\ &\qquad - \int_0^T V^*(t, -p'(t))\, dt - l(p(0)). \end{aligned} \tag{2.3}$$

Let us put, for $p \in X^d$,

$$J_D(p, d) = -\int_0^T L^*(t, p(t) + d)\, dt + \int_0^T V^*(t, -p'(t))\, dt + l(p(0)).$$

From (2.3) we infer for $p \in X^d$ that

$$\sup_{x \in X} -J^{\#}_x(-p, -d_p) = -J_D(p, d_p). \tag{2.4}$$

We can also define a type of the second conjugate of J: for $y \in L^2$, $a \in R^n$, $x \in X$, $p \in X^d$, put

$$J_x^{\#\#}(y, a) = \sup_{p \in X^d} \left\{ \int_0^T \langle y(t), -p'(t) \rangle dt + \int_0^T \langle x(t) + c_x, -p'(t) \rangle dt - \int_0^T L(t, x'(t))\, dt - \int_0^T V^*(t, -p'(t))\, dt \right\} + \inf_{d \in R^n} \{ \langle a, d \rangle + \langle x(T), d \rangle \}.$$

We assert that $J_x^{\#\#}(0,0) = -J(x, c_x)$. To prove this, we use the special structure of X. First we observe that for each $x \in X$ there exists $\bar{p} \in X^d$ such that $\bar{p}'(\cdot) = -V_x(\cdot, x(\cdot) + c_x)$, and therefore

$$\int_0^T \langle -\bar{p}'(t), x(t) + c_x \rangle dt - \int_0^T V^*(t, -\bar{p}'(t))\, dt = \int_0^T V(t, x(t) + c_x)\, dt.$$

Next let us note that

$$\begin{aligned} &\int_0^T \langle -\bar{p}'(t), x(t) + c_x \rangle dt - \int_0^T V^*(t, -\bar{p}'(t))\, dt \\ &\quad \le \sup_{p \in X^d} \left\{ \int_0^T \langle -p'(t), x(t) + c_x \rangle dt - \int_0^T V^*(t, -\bar{p}'(t))\, dt \right\} \\ &\quad = \sup_{p' \in L^2} \left\{ \int_0^T \langle -p'(t), x(t) + c_x \rangle dt - \int_0^T V^*(t, -\bar{p}'(t))\, dt \right\} \\ &\quad = \int_0^T V(t, x(t) + c_x)\, dt. \end{aligned}$$

Hence we see that, for $x \in X$,

$$\begin{aligned} J_x^{\#\#}(0,0) &= -\int_0^T (-V(t, x(t) + c_x) + L(t, x'(t)))\, dt - l(x(T)) \\ &= -J(x, c_x). \end{aligned} \tag{2.5}$$

We easily compute (see (2.4))

$$\begin{aligned} \sup_{x \in X} J_x^{\#\#}(0,0) &= \sup_{x \in X} \sup_{p \in X^d} -J_x^{\#}(-p, -d_p) \\ &= \sup_{p \in X^d} \sup_{x \in X} -J_x^{\#}(-p, -d_p) \\ &= \sup_{p \in X^d} -J_D(p, d_p) = \sup_{p \in X^d} \inf_{d} -J_D(p, d), \end{aligned} \tag{2.6}$$

where the last equality is a consequence of the following lemma

LEMMA 2.1. *For any $p \in X^d$ which correspond to $x \in X$ with $x(0) = x(T) = 0$ the constant d_p from the specification of X^d is a minimizer of the functional*

$$d \to \int_0^T L^*(t, p(t) + d)\, dt.$$

Proof. We observe, from the definition of X^d, that $p(t) + d_p = L_{x'}(t, x'(t))$ a.e. in $[0, T]$ for some $x \in X$. This means that $x'(t) = L_p^*(t, p(t) + d_p)$ a.e. in $[0, T]$. Integrating this equality yields, since x is periodic and L^* convex, the assertion of the lemma.

We shall need one more

LEMMA 2.2. *For any $p \in A$ such that $p'(t) = -V_x(t, x(t) + c_x)$, for $x \in X$, and c_x being a minimizer of the functional $c \to \int_0^T V(t, x(t) + c)\, dt$, we have $p(0) = p(T)$.*

Proof. The assumption of the lemma yields that $\int_0^T V_x(s, x(s) + c_x)\, ds = 0$, and the proof is completed.

Hence, from the above lemmas and (2.6) we obtain the following duality principle:

THEOREM 2.1. *For functionals J and J_D we have the duality relation*

$$\inf_{x \in X} \sup_c J(x, c) = \inf_{p \in X^d} \sup_d J_D(p, d). \tag{2.7}$$

Denote by $\partial J_x(y, a)$ the subdifferential of J_x with respect to the first variable. In particular, if $1/q + 1/q' = 1$ then

$$\partial J_{\bar{x}}(0, 0) = \left\{ q \in L^{q'} : \int_0^T V^*(t, q(t))\, dt + \int_0^T V(t, \bar{x}(t) + c_{\bar{x}})\, dt = \int_0^T \langle q(t), \bar{x}(t) + c_{\bar{x}} \rangle dt \right\}.$$

The next result formulates a variational principle for "minmax" arguments.

THEOREM 2.2. *Let $\bar{x} \in X$ be such that*

$$+\infty > J(\bar{x}, c_{\bar{x}}) = \inf_{x \in X} \sup_c J(x, c) > -\infty,$$

and let the set $\partial J_{\bar{x}}(0, 0)$ be nonempty. Then there exists $-\bar{p}' \in \partial J_{\bar{x}}(0, 0)$ with $\bar{p}(t) = -\int_t^T \bar{p}'(s)\, ds$ belonging to X^d, such that $\bar{p}$ together with $d_{\bar{p}}$ satisfies

$$J_D(\bar{p}, d_{\bar{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d).$$

Furthermore,

$$J_{\bar{x}}(0, 0) + J_{\bar{x}}^{\#}(-\bar{p}, -d_{\bar{p}}) = 0, \tag{2.8}$$

$$J_D(\bar{p}, d_{\bar{p}}) - J_{\bar{x}}^{\#}(-\bar{p}, -d_{\bar{p}}) = 0. \tag{2.9}$$

Proof. By Theorem 2.1, to prove the first assertion it suffices to show that $J(\bar{x}, c_{\bar{x}}) \geq J_D(\bar{p}, d_{\bar{p}})$. We note, from the form of $J(x)$ and the finiteness of $J(\bar{x}, c_{\bar{x}})$, that $\bar{x}(T) = \bar{x}(0) = 0$. Let us observe that $-\bar{p}' \in \partial J_{\bar{x}}(0, 0)$ means, in fact, that $-\bar{p}'(t) = V_x(t, \bar{x}(t) + c_{\bar{x}})$, a.e. $t \in [0, T]$. By Lemma 2.2 each primitive of $\bar{p}'$ is a periodic function. Since $\bar{x} \in X$, there exists an $\tilde{x} \in X$ such that $\bar{p}(t) = \int_t^T V_x(s, \bar{x}(s) + c_{\bar{x}})\, ds = \int_t^T -\frac{d}{ds} L_{x'}(s, \tilde{x}'(s))\, ds = L_{x'}(t, \tilde{x}'(t)) - L_{x'}(T, \tilde{x}'(T))$. Putting $d_{\bar{p}} = L_{x'}(T, \tilde{x}'(T))$, we have that $\bar{p}(t) = L_{x'}(t, \tilde{x}'(t)) - d_{\bar{p}}$ belongs to X^d. Hence, we have

$$
\begin{aligned}
-J(\bar{x}, c_{\bar{x}}) &= \int_0^T (V(t, \bar{x}(t) + c_{\bar{x}}) - L(t, \bar{x}'(t)))\, dt \\
&= \int_0^T (-V^*(t, -\bar{p}'(t)) - L(t, \bar{x}'(t)))\, dt \\
&\quad + \int_0^T \langle \bar{x}(t) + c_{\bar{x}}, -\bar{p}'(t) \rangle dt \\
&\leq \int_0^T (-V^*(t, -\bar{p}'(t)) + L^*(t, \bar{p}(t) + d_{\bar{p}}))dt = -J_D(\bar{p}, d_{\bar{p}}).
\end{aligned}
$$

Therefore, $J(\bar{x}, c_{\bar{x}}) \geq J_D(\bar{p}, d_{\bar{p}})$, and since, by Lemma 2.1 $d_{\bar{p}}$ is a minimizer of the functional $d \to \int_0^T L^*(t, \bar{p}(t) + d)\, dt$, we also have $J(\bar{x}, c_{\bar{x}}) = J_D(\bar{p}, d_{\bar{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d)$. Thus the first assertion is proved.

The second assertion is a simple consequence of two facts: $J_{\bar{x}}(0, 0) = -J(\bar{x}, c_{\bar{x}})$, so $J_{\bar{x}}(0, 0) + J(\bar{x}, c_{\bar{x}}) = 0$ and $-\bar{p}' \in \partial J_{\bar{x}}(0, 0)$, i.e., $J_{\bar{x}}(0, 0) + J_{\bar{x}}^{\#}(-\bar{p}, -d_{\bar{p}}) = 0$, and so equality (2.8). Then we get equality (2.9) by joining the last equality and the equality $J(\bar{x}, c_{\bar{x}}) = J_D(\bar{p}, d_{\bar{p}})$.

From equations (2.8) and (2.9) we are able to derive a dual to (1.1) Euler–Lagrange equations.

COROLLARY 2.1. *Let $\bar{x} \in X$ be such that*

$$
+\infty > J(\bar{x}, c_{\bar{x}}) = \inf_{x \in X} \sup_c J(x, c) > -\infty.
$$

Then there exists $\bar{p} \in X^d$ such that the pair $(\bar{x}, \bar{p})$ satisfies the relations

$$
-\bar{p}'(t) = V_x(t, \bar{x}(t) + c_{\bar{x}}), \tag{2.10}
$$

$$
\bar{p}(t) + d_{\bar{p}} = L_{x'}(t, \bar{x}'(t)), \tag{2.11}
$$

$$
J_D(\bar{p}, d_{\bar{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d) = \inf_{x \in X} \sup_c J(x, c) = J(\bar{x}, c_{\bar{x}}). \tag{2.12}
$$

Proof. By the assumptions on V we see that $y \to \int_0^T V(t, y(t))\, dt$ is finite in L^q, convex, and lower semicontinuous. Therefore $J_{\bar{x}}(0, y)$ is continuous in L^q. Hence $\partial J_{\bar{x}}(0, 0)$ is nonempty, and so the existence of $\bar{p}'$ in

Theorem 2.2 is now obvious. Equations (2.8) and (2.9) imply

$$\int_0^T V(t, \bar{x}(t) + c_{\bar{x}})\, dt + \int_0^T V^*(t, -\bar{p}'(t))\, dt$$
$$- \int_0^T \langle \bar{x}(t) + c_{\bar{x}}, -\bar{p}'(t)\rangle dt = 0,$$
$$\int_0^T L^*(t, \bar{p}(t) + d_{\bar{p}})\, dt + \int_0^T L(t, \bar{x}'(t))\, dt$$
$$- \int_0^T \langle \bar{x}'(t), \bar{p}(t) + d_{\bar{p}}\rangle dt = 0,$$

and then (2.10) and (2.11). Relations (2.12) are a direct consequence of Theorem 2.1 and Theorem 2.2.

As a direct consequence of the above corollary and the definition of X^d we have

COROLLARY 2.2. *By the same assumptions as in Corollary* 2.1 *there exists a pair* $(\bar{x}, \bar{p}) \in X \times X^d$ *satisfying, together with* $(c_{\bar{x}}, d_{\bar{p}})$, *relations* (2.12), *and the pair* $(\bar{x} + c_{\bar{x}}, \bar{p} + d_{\bar{p}})$ *is a solution to* (1.1). *Conversely, each pair* $(\bar{x}, \bar{p})$ *satisfying, together with* $(c_{\bar{x}}, d_{\bar{p}})$, *relations* (2.12) *also satisfies Eqs.* (2.10) *and* (2.11).

3. VARIATIONAL PRINCIPLES AND A DUALITY GAP FOR MINIMIZING SEQUENCES

In this section we show that a statement similar to Theorem 2.2 is true for a minimizing sequence of J.

THEOREM 3.1. *Let* $\{(x_j, c_{x_j})\}$, $x_j \in X$, $j = 1, 2, \ldots,$ *be a minimizing sequence for J and let*

$$+\infty > J(x_j, c_{x_j}) > -\infty, \qquad j = 1, 2, \ldots .$$

Then there exist $-p_j' \in \partial J_{x_j}(0, 0)$ *with* $p_j \in X^d$, *such that* $\{(p_j, d_{p_j})\}$ *is a minimizing sequence for* J_D, *i.e.*,

$$\inf_{x_j \in X} J(x_j, c_{x_j}) = \inf_{x_j \in X} \sup_{c \in R^n} J(x_j, c) = \inf_{p_j \in X^d} \sup_{d \in R^n} J_D(p_j, d)$$
$$= \inf_{p_j \in X^d} J_D(p_j, d_{p_j}).$$

Furthermore,

$$J_{x_j}(0, 0) + J^{\#}_{x_j}(-p_j, -d_{p_j}) = 0,$$
$$J_D(p_j, d_{p_j}) - J^{\#}_{x_j}(-p_j, -d_{p_j}) \le \varepsilon,$$
$$0 \le J(x_j, c_{x_j}) - J_D(p_j, d_{p_j}) \le \varepsilon,$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

Proof. We have that $\infty > \inf_{x_j \in X} J(x_j, c_{x_j}) = a > -\infty$, and therefore we may assume $x_j(0) = x_j(T)$. Thus, for a given $\varepsilon > 0$ there exists j_0 such that $J(x_j, c_{x_j}) - a < \varepsilon$, for all $j \geq j_0$. Furthermore, the proof is similar to that of Theorem 2.2, so we only sketch it. First we observe that $\partial J_{x_j}(0,0)$ is nonempty for $j \geq j_0$, and $-p_j' \in \partial J_{x_j}(0,0)$ implies that $\int_0^T p_j'(t)\,dt = 0$. Accordingly, from the definition of X^d let us take as a primitive of p_j' such p_j that $p_j(T) = 0$ and in fact $p_j(0) = 0$. Therefore, for all $d \in R^n$, we also have

$$\begin{aligned} -J(x_j, c_{x_j}) &= \int_0^T (V(t, x_j(t) + c_{x_j}) - L(t, x_j'(t)))\,dt \\ &= \int_0^T (-V^*(t, -p_j'(t)) - L(t, x_j'(t)))\,dt \\ &\quad + \int_0^T \langle x_j(t) + c_{x_j}, -p_j'(t)\rangle dt \\ &\leq \int_0^T (-V^*(t, -p_j'(t)) + L^*(t, p_j(t) + d))\,dt \\ &= -J_D(p_j, d). \end{aligned}$$

Hence, due to Theorem 2.1,

$$a + \varepsilon \geq \sup_{d \in R^n} J_D(p_j, d) = J_D(p_j, d_{p_j}) \geq a, \qquad \text{for } j \geq j_0.$$

The second assertion is a simple consequence of two facts: $J_{x_j}(0,0) = -J(x_j, c_{x_j})$, so $J_{x_j}(0,0) + J(x_j, c_{x_j}) = 0$ and $-p_j' \in \partial J_{x_j}(0,0)$, i.e., $J_{x_j}(0,0) + J_{x_j}^{\#}(-p_j, -d_{p_j}) = 0$.

A direct consequence of this theorem is the following corollary.

COROLLARY 3.1. *Let $\{(x_j, c_{x_j})\}$, $x_j \in X$, $j = 1, 2, \ldots$, be a minimizing sequence for J and let*

$$+\infty > J(x_j, c_{x_j}) > -\infty, \qquad j = 1, 2, \ldots .$$

If

$$-p_j'(t) = V_x(t, x_j(t) + c_{x_j}),$$

then $p_j(t) = -\int_t^T p_j'(s)\,ds$ belongs to X^d, and $\{(p_j, d_{p_j})\}$ is a minimizing sequence for J_D, i.e.,

$$\begin{aligned} \inf_{x_j \in X} J(x_j, c_{x_j}) &= \inf_{x_j \in X} \sup_{c \in R^n} J(x_j, c) = \inf_{p_j \in X^d} \sup_{d \in R^n} J_D(p_j, d) \\ &= \inf_{p_j \in X^d} J_D(p_j, d_{p_j}). \end{aligned}$$

Furthermore,

$$\begin{aligned} J_D(p_j, d_{p_j}) - J^{\#}_{x_j}(-p_j, -d_{p_j}) &\leq \varepsilon, \\ 0 \leq J(x_j, c_{x_j}) - J_D(p_j, d_{p_j}) &\leq \varepsilon, \end{aligned} \tag{3.1}$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

4. THE EXISTENCE OF "MAXMIN"

The last problem which we have to solve is to prove the existence of $\bar{x} \in X$ such that

$$J(\bar{x}, c_{\bar{x}}) = \min_{x \in X} \max_{c \in R^n} J(x, c).$$

To obtain this it is enough to use hypothesis (H1), the results of the former section, and known compactness theorems.

THEOREM 4.1. *Under hypothesis* (H1) *there exists* $\bar{x} \in X$ *such that* $J(\bar{x}, c_{\bar{x}}) = \min_{x \in X} \max_{c \in R^n} J(x, c)$.

Proof. Let us observe that by (H1) for each $x \in X$ there exists c_x such that $\max_{c \in R^n} J(x, c) = J(x, c_x)$. Next we show that $J(x, c_x)$ is bounded below on X. By (1.3), (1.4), and the properties of c_x we obtain

$$\begin{aligned} J(x, c_x) &\geq \frac{\alpha_1}{2} \int_0^T |x'(t)|^2 dt - \frac{\beta_2}{q} \|x\|^q_{L^q} + d_1 - k_2 \\ &\geq \frac{\alpha_1}{2} \|x'\|^2 - \frac{\alpha_1}{4} \|x'\|^2 + d_1 - k_2. \end{aligned} \tag{4.1}$$

From (4.1) we infer the boundedness below of J on X as well as that the sets $S_b = \{(x, c_x) : x \in X, c_x \in R^n, J(x, c_x) \leq b\}$, $b \in R$, are nonempty for sufficiently large b and bounded with respect to the norm $|c_x| + \|x'\|_{L^2}$. The last means that S_b, $b \in R$ are relatively weakly compact in $A_0 \oplus R^n$. It is a well-known fact that the functional J is weakly lower semicontinuous in $A_0 \oplus R^n$ and thus also in $X \oplus R^n$. Therefore there exists a sequence $\{x_n\}$, $x_n \in X$, such that $x_n \rightharpoonup \bar{x}$ weakly in A_0 with $\bar{x} \in A_0$, together with $c_{x_n} \to c_{\bar{x}} \in R^n$, and $\liminf_{n \to \infty} J(x_n, c_{x_n}) \geq J(\bar{x}, c_{\bar{x}})$. Moreover, we know that $\{(x_n, c_{x_n})\}$ is uniformly convergent to $(\bar{x}, c_{\bar{x}})$. To finish the proof we must show only that $\bar{x} \in X$.

To prove this we apply the duality results of Section 3. To this end let us recall from Corollary 3.1 that for

$$p_n'(t) = -V_x(t, x_n(t) + c_{x_n}) \tag{4.2}$$

$p_n(t) = -\int_t^T p_n'(s)\, ds$ belongs to X^d and take d_{p_n} such that $J_D(p_n, d_{p_n}) = \max_{d \in R^n} J_D(p_n, d)$. Then $\{(p_n, d_{p_n})\}$ is a minimizing sequence for J_D.

We easily check that $\{d_{p_n}\}$ is a bounded sequence, and therefore we may assume (up to a subsequence) that it is convergent. From (4.2) we infer that $\{p'_n\}$ is a bounded sequence in a L^2 norm and that it is pointwise convergent to

$$\bar{p}'(t) = -V_x(t, \bar{x}(t) + c_{\bar{x}}), \tag{4.3}$$

and so $\{p_n\}$ is uniformly convergent to $\bar{p}$, where $\bar{p}(t) = -\int_t^T \bar{p}'(s)\,ds$. We can choose $d_{\bar{p}}$ satisfying the equality $\max_{d\in R^n} J_D(\bar{p}, d) = J_D(\bar{p}, d_{\bar{p}})$.

By Corollary 3.1 (see (3.1)) we also have (taking into account (4.2)) that for $\varepsilon_n \to 0$ $(n \to \infty)$

$$\begin{aligned} 0 \leq \int_0^T (L^*(t, p_n(t) + d_{p_n}) + L(t, x'_n(t)))\,dt \\ - \int_0^T \langle x'_n(t), p_n(t) + d_{p_n}\rangle dt \leq \varepsilon_n, \end{aligned}$$

and so, taking a limit,

$$0=\int_0^T L^*(t,\bar{p}(t)+d_{\bar{p}})dt+\lim_{n\to\infty}\int_0^T L(t,x'_n(t))dt-\int_0^T \langle \bar{x}'(t),\bar{p}(t)+d_{\bar{p}}\rangle dt,$$

and next, in view of the property of Fenchel inequality,

$$0=\int_0^T L^*(t,\bar{p}(t)+d_{\bar{p}})dt+\int_0^T L(t,\bar{x}'(t))dt-\int_0^T \langle \bar{x}'(t),\bar{p}(t)+d_{\bar{p}}\rangle\,dt. \tag{4.4}$$

Applying now Ekeland's variational principle [4] to the ϵ-subdifferential of $\int_0^T L(t, x'_n(t))\,dt$ at $x'_n(\cdot)$, we deduce that $\{x'_n\}$ is strongly convergent in L^2 to $\bar{x}'$. Therefore, as $x_n \in \overline{X}$, $\bar{x}$ satisfies $\beta_2/q\|\bar{x}\|_{L^q}^q \leq \alpha_1/4\|\bar{x}'\|_{L^2}^2$, and so it belongs to $\overline{X}$. From (4.4) we also have

$$\bar{p}(t) + d_{\bar{p}} = L_{x'}(t, \bar{x}'(t)). \tag{4.5}$$

Joining (4.3) and (4.5), we get

$$\frac{d}{dt}L_{x'}(t, \bar{x}'(t)) = -V_x(t, \bar{x}(t) + c_{\bar{x}}).$$

The last means that $\bar{x} \in X$, and so the proof is completed.

A direct consequence of Theorem 4.1 and Corollary 2.2 is the following main theorem.

THEOREM 4.2. *Assume hypotheses* (H) *and* (H1) *and that X contains at least one element x with $x(0) = 0 = x(T)$. Then there exists a pair* $(\bar{x} + c_{\bar{x}}, \bar{p} + d_{\bar{p}})$ *that is a solution to* (1.1) *and such that*

$$J(\bar{x}, c_{\bar{x}}) = \min_{x\in X}\max_{c\in R^n} J(x, c) = \min_{p\in X^d}\max_{d\in R^n} J_D(p, d) = J_D(\bar{p}, d_{\bar{p}}).$$

5. EXAMPLE

Consider the problem

$$x''(t) + W_x(t, x(t)) = 0, \qquad \text{a.e. in } R$$

$$x(0) = x(T), \qquad x'(0) = x'(T),$$

where $W(\cdot, x)$ is a T-periodic, measurable function in R, and $W(t, \cdot)$ is a convex, Gateaux differentiable function satisfying the following growth conditions: there exist $0 < \beta_1 < \beta_2$, $q_1 > 1$, $q > 2$, $k_1, k_2 \in R$ such that for $x \in L^q$

$$k_1 + \frac{\beta_1}{q_1}\|x\|_{L^{q_1}}^{q_1} \le \int_0^T W(t, x(t))\, dt \le \frac{\beta_2}{q}\|x\|_{L^q}^q + k_2.$$

In the notation of the paper we have $L(t, x') = \frac{1}{2}|x'|^2$, and $V(t, x) = W(t, x)$. It is easily seen that assumptions (H) and (H1), with $\alpha_1 = 1$, are satisfied. Therefore what we have to do is to construct a nonempty set X.

We shall show that the set $X = \overline{X} \cap \{x \in A_0 : \|x\|_{L^q} < 1\}$ is the set X which we are looking for. For the convenience of calculation we assume that $T = 1$. We must prove that if $x \in X$ then the function

$$t \to \int_0^t \int_0^s W_x(\tau, x(\tau) + c_x)\, d\tau\, ds + at = w(t) + at \tag{5.1}$$

also belongs to X for some $a \in R^n$. It is enough to show that there exists an $\bar{a} \in R^n$ such that $t \to w(t) + \bar{a}t = \bar{w}(t)$ belongs to $\overline{X}$ and that $\|\bar{w}\|_{L^q} < 1$. Since $T = 1$ we have $\|w\|_{L^q}^2 \le \|w'\|_{L^2}^2$, and because $q > 2$ all we have to do is prove that $\|\bar{w}\|_{L^q}^2 < \gamma = \min\{1, (q/4\beta_2)^{2/(q-2)}\}$.

To this end let us calculate

$$\min_a \int_0^1 |w'(t) + a|^2\, dt.$$

Since it is a convex function of variable a which satisfies the growth condition, we easily see that the minimum is attained for

$$\bar{a} = -w(1).$$

Therefore let us calculate

$$\int_0^1 |w'(t) - w(1)|^2\, dt = \int_0^1 |w'(t)|^2\, dt - |w(1)|^2.$$

We additionally assume that

(HE). *W has the property that for each $x \in A_0$ with $\|x\|_{L^q}^2 < 1$, the relevant w (see (5.1)) satisfies*

$$\int_0^1 |w'(t)|^2 dt < |w(1)|^2 + \gamma.$$

The last inequality implies that $\|\bar{w}\|_{L^q}^2 < \gamma$, and so the proof is completed.

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